Level $\Gamma_0(p)$

Adelic double quotient

$$Y_K = GL_2(\mathbb{Q}) \backslash \left(GL_2(\mathbb{A}_f)/K \times \mathcal{H}^{\pm} \right)$$

$K \subset GL_2(\mathbb{A}_f)$ open compact "level"

We saw After conjugation, $K \subseteq GL_2(\hat{\mathbb{Z}})$
(use compact)

$\exists\, N \geq 1$ s.th. $K(N) \subseteq K$ (use openness)

$$GL_2(\hat{\mathbb{Z}}) = \varprojlim_N GL_2(\mathbb{Z}/N) \underset{CRT}{=} \prod_p GL_2(\mathbb{Z}_p)$$

Tautology $K = \{ \gamma \in GL_2(\hat{\mathbb{Z}}) \mid$
$[\gamma \bmod N] \in K/K(N)$
$\subseteq GL_2(\mathbb{Z}/N) \}$

Often $K = \prod_p K_p \qquad K_p \subseteq GL_2(\mathbb{Q}_p)$

Typical choices $K(N) = \left\{ \gamma \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \bmod N \right\}$

$\subseteq K_1(N) = \left\{ \gamma \equiv \begin{pmatrix} 1 & * \\ & * \end{pmatrix} \bmod N \right\}$

$\subseteq K_0(N) = \{ \gamma \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \}$

<u>Their lattice interpretations</u>

$V = \mathbb{Q}_p^2$ <u>lattice</u> $\Lambda \subset V$ $\underset{\text{def}}{=}$ $\mathbb{Z}_p$-submodule $\cong \mathbb{Z}_p^2$.

$GL_2(\mathbb{Q}_p) \curvearrowright \{ \Lambda \subset V \}$ transitively

(E.g. Given Λ, pick $\mathbb{Z}_p$-basis λ_1, λ_2

Then $\exists g$ s.th. $g(e_i) = \lambda_i$

Then $g\,\mathbb{Z}_p^2 = \Lambda$.)

$GL_2(\mathbb{Z}_p)$ is stabilizer of $\mathbb{Z}_p^2$

$$\Longrightarrow \quad GL_2(\mathbb{Q}_p)/GL_2(\mathbb{Z}_p) \xrightarrow{\cong} \{ \Lambda \subset V \}$$
$$g \longmapsto g \cdot \mathbb{Z}_p^2.$$

<u>Lem</u>

$$GL_2(\mathbb{Q}_p)/K(p^n) \xrightarrow{\cong} \{ \Lambda \subset V + \lambda_1, \lambda_2 \ \mathbb{Z}/p^n\text{-basis of } \Lambda/p^n\Lambda \}$$

$$GL_2(\mathbb{Q}_p)/K_1(p^n) \xrightarrow{\cong} \{ \Lambda \subset V + \lambda \text{ non-torsion } \in \Lambda/p^n\Lambda.$$

$$GL_2(\mathbb{Q}_p)/K_0(p^n) \xrightarrow{\cong} \{\Lambda \subset V + C \subset \Lambda/p^n\Lambda \text{ cyclic order } p^n\}$$

we obtain E/k EC, $k=\bar{k}$, char $k \neq p$

$$\underbrace{\mathrm{Isom}_{\mathbb{Z}_p}(\mathbb{Z}_p^2, T_p(E))}_{X} \circlearrowleft GL_2(\mathbb{Z}_p)\text{-PHS},$$

action from right.

$$X/K(p^n) \xrightarrow{\cong} \{\text{level-}p^n\text{-str on } E\}$$

$$\eta \longmapsto \big((\eta(e_1) \bmod p^n), \eta(e_2) \bmod p^n\big)$$

$$X/K_1(p^n) \xrightarrow{\cong} \{x \in E[p^n] \text{ of order } p^n\}$$

$$\eta \longmapsto \eta(e_1 \bmod p^n)$$

$$X/K_0(p^n) \xrightarrow{\cong} \{C \subset E[p^n] \text{ cyclic of order } p^n\}$$

$$\eta \longmapsto \langle \eta(e_1 \bmod p^n)\rangle.$$

§ An example $N \geq 3$, $p \nmid N$

$$K(N,p) := K(N) \cap K_0(p) \subseteq GL_2(\hat{\mathbb{Z}}).$$

<u>Motivation</u> Have $-1 \in K_0(p)$

$\longleftrightarrow$ $-1 \in \mathrm{Aut}(E, C \subset E)$ subgroup

So cannot obtain fine moduli for $Y_{K_0(p)}$, have to add auxiliary extra level str.

<u>Note</u> $K(N,p) = \prod_{\ell \mid N} K(\ell^{v_\ell(N)}) \times K_0(p) \times \prod_{\ell \nmid pN} GL_2(\mathbb{Z}_\ell)$

<u>Lem</u> $Y_{K(N,p)} = \{(E, \alpha, C)/\mathbb{C}\}/\cong$

·) E EC

·) $\alpha : \mathbb{Z}/N^2 \xrightarrow{\cong} E[N]$ level str

·) $C \subseteq E$ ~~cyclic~~ order p

·) $(E, \alpha, C) \cong (E', \alpha', C')$ $\overset{=}{\text{def}}$

$\exists\, \gamma : E \to E'$ s.t. $\gamma \circ \alpha = \alpha'$, $\gamma(C) = C'$

Proof $\{(E,\alpha,C)\}/\cong$ $\qquad K := K(N,p)$

$$\xrightarrow{\cong} \; \mathrm{Gl}_2(\mathbb{Z}) \backslash \{(E, (\tau_1,\tau_2), \eta)\}/\cong / K$$

·) $\tau_1, \tau_2 \in \pi_1(E,e)$ basis

·) $\eta : \hat{\mathbb{Z}}^2 \xrightarrow{\cong} T(E) := \prod_p T_p(E)$ "full level str"

·) $\mathrm{Gl}_2(\mathbb{Z})$ acts as $(\tau_1,\tau_2)\circ\gamma^t$

·) K acts as $\eta\circ g$

·) Map Given (E,α,C), pick any τ_1,τ_2

pick any η s.th. $\eta \bmod N = \alpha$

$\langle \eta(e_1) \bmod p\rangle = C$

$$\xrightarrow{\cong} \; \mathrm{Gl}_2(\mathbb{Z}) \backslash (\mathrm{Gl}_2(\hat{\mathbb{Z}})/K \times \mathcal{H}^{\pm})$$

by $(E, (\tau_1,\tau_2), \eta) \longmapsto (\hat{\tau}^{-1}\circ\eta, \tau_1/\tau_2)$

Explanation (τ_1,τ_2) provide full level str

$\hat{\tau} : \hat{\mathbb{Z}}^2 \longrightarrow T(E), \quad e_i \longmapsto \left(\frac{\tau_i}{n}\right)_{n\geq 1}$

Then η & τ differ by unique $h \in \mathrm{Gl}_2(\hat{\mathbb{Z}})$

$$\begin{tikzcd} \hat{\mathbb{Z}}^2 \arrow[r, "\eta"] \arrow[d] & T(E) \\ \hat{\mathbb{Z}}^2 \arrow[ur, "\hat{\tau}"'] & \end{tikzcd}$$

$h = \hat{\tau}^{-1} \circ \eta$

This map provides a bijection

$$\{ (E, (\tau_1, \tau_2), \eta) \}/\cong \xrightarrow{\cong} GL_2(\hat{\mathbb{Z}}) \times \mathcal{H}^{\pm}$$

$GL_2(\mathbb{Z}) \times K$ - action on RHS given by

$$\gamma \cdot (h, \tau) \cdot g = ({}^t\gamma^{-1} \cdot h \cdot g, \gamma\tau)$$

$$\xrightarrow{\cong} GL_2(\mathbb{Q}) \backslash \left(GL_2(\mathbb{A}_f)/K \times \mathcal{H}^{\pm} \right)$$

by $[h, \tau] \longmapsto [h, \tau]$ $\left(\text{or} \mapsto [{}^t h^{-1}, \tau] \right)$

since ${}^t K^{-1} = K$

<u>Injective</u> $(h_1, \tau_1) = (\gamma h_2 g, \gamma \tau_2)$

s.th. $h_i \in GL_2(\hat{\mathbb{Z}})$, $\gamma \in GL_2(\mathbb{Q})$, $g \in K$

$\Longrightarrow \gamma = h_1 g^{-1} h_2^{-1} \in GL_2(\hat{\mathbb{Z}})$

$\Longrightarrow \gamma \in GL_2(\mathbb{Q}) \cap GL_2(\hat{\mathbb{Z}}) = GL_2(\mathbb{Z})$

<u>Surjectivity</u> Given $[h,\tau]$, use class number 1 property: $GL_2(\mathbb{A}_f) = GL_2(\mathbb{Q}) \cdot GL_2(\hat{\mathbb{Z}})$

So can write $h = \gamma \cdot h_0$

Then $[h,\tau] = [h_0, \gamma^{-1}\tau]$. $\square$

§ <u>Integral models for $K(N,p)$</u>

$N \geq 3$, $p \nmid N$

$$M_{N,p} : (Sch/_{\mathbb{Z}[N^{-1}]})^{op} \longrightarrow (Set)$$
$$S \longmapsto \{(E,\alpha,C)\}/_{\cong}$$

.) E/S EC

.) $\alpha : \underline{\mathbb{Z}/N}^2_S \xrightarrow{\cong} E[N]$

.) $C \hookrightarrow E$ closed subgroup scheme finite loc free of rank p /S.

<u>Rmk</u> More complicated for $K_0(p^n)$ b/c then would need notion <u>cyclic</u> order p^n group. (Use Drinfeld level structures.)

Comes with map $\mathcal{M}_{N,p} \to \mathcal{M}_N$

$$(E, \alpha, C) \mapsto (E, \alpha)$$

<u>Thm</u> $\mathcal{M}_{N,p}$ representable by an affine scheme.
It is regular + finite flat of degree $p+1$ over $\mathcal{M}_N$.

<u>Representability</u> Recall: Fin loc free rank r group sch $/S$ =

$\mathcal{A}/\mathcal{O}_S$ loc free rank r $/\mathcal{O}_S$

+ $1 : \mathcal{O}_S \to \mathcal{A}$ unit
+ $m : \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$ multiplication
+ $1^* : \mathcal{A} \to \mathcal{O}_S$ counit
+ $m^* : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}$ comultiplicat.

<u>s.th.</u> Hopf-alg axioms hold.

<u>Prop</u> G/S rank r fin loc free grp sch

$$\mathrm{Sub}_d(G) : \mathrm{Sch}/S^{op} \to \mathrm{Sets}$$
$$T \mapsto \{ H \subset G_T, \text{ fin loc free rank } d \}$$

subgroup

is representable by proj S-scheme.

Proof $G = \underline{\mathrm{Spec}}_{\mathcal{O}_S}\, \mathcal{A}$ $\qquad D := Gr_d(\mathcal{A})$ $\qquad$ dim $= d(r-d)$

$$D(u : T \to S) = \left\{ u^*\mathcal{A} \twoheadrightarrow Q \,,\; Q \text{ loc free rank } d / \mathcal{O}_T \right\}$$

$q : D \to S$ projective, locally $\cong Gr(r,d)_S$.

$q^*\mathcal{A} \to Q$ universal quotient

$I :=$ kernel

Unit $1 : \mathcal{O}_D \to q^*\mathcal{A} \to Q$

Multiplication

$$\begin{tikzcd}
0 \arrow[r] & \underset{(r-d)r}{I \otimes q^*\mathcal{A} + q^*\mathcal{A} \otimes I} \arrow[r] \arrow[drr, bend right, "f"'] & q^*(\mathcal{A} \otimes \mathcal{A}) \arrow[r] \arrow[d, "m"] \arrow[dr] & Q \otimes Q \arrow[r] \arrow[d, dashed, "?x"] & 0 \\
& & q^*\mathcal{A} \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

$dr(r-d) \to f$ Equations $\qquad$ dim d

Counit

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[dr, "g"'] & q^*\mathcal{A} \arrow[r] \arrow[d, "1^*"] & Q \arrow[r] \arrow[dl, dashed, "?y"] & 0 \\
& & \mathcal{O}_D & &
\end{tikzcd}$$

Comultiplication

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[dr, bend right, "h"'] & q^*\mathcal{A} \arrow[r] \arrow[d, "m^*"] & Q \arrow[r] \arrow[d, dashed, "?\, z"] & 0 \\
& & q^*(\mathcal{A}\otimes\mathcal{A}) \arrow[r] & Q\otimes Q \arrow[r] & 0
\end{tikzcd}$$

Consider $u : T \to S$, $u^*\mathcal{A} \to B$

$\longleftrightarrow$ $b : T \to D$

B is an algebra i.e. defines closed subscheme

$$\underline{\mathrm{Spec}}_{\mathcal{O}_T} B \longrightarrow \underline{\mathrm{Spec}}_{\mathcal{O}_T}\, u^*\mathcal{A}$$

$\Longleftrightarrow$ $b^*f = 0$ (i.e. x exists)

+ contains unit section $\Longleftrightarrow$ $b^*g = 0$
(i.e. y exists)

+ subgroup scheme $\Longleftrightarrow$ $b^*h = 0$
(i.e. z exists.)

$\Longrightarrow$ $\mathrm{Sub}_d(G) \cong V(f, g, h) \subseteq D$ $\square$

Representability of $\mathcal{M}_{N,p}$

Consider universal EC $(\mathcal{E}, \alpha)/\mathcal{M}_N$

$\mathcal{E}[p] \to \mathcal{M}_N$ fin loc free rank p^2 group scheme

Then $\mathcal{M}_{N,p} = \mathrm{Sub}_p(\mathcal{E}[p])$ $\square$

$(E, \alpha, C) \in \mathcal{M}_{N,p}(S)$

$\exists!$ $u : S \to \mathcal{M}_N$ s.th. $(E, \alpha) \cong u^*(\mathcal{E}, \alpha^{univ})$

Via u, view S as $\mathcal{M}_N$-scheme

$$\underbrace{C \subseteq E[p] = u^* \mathcal{E}[p]}_{c}$$

$$\begin{tikzcd} & \mathcal{M}_{N,p} \arrow[d] \\ S \arrow[ur, "\exists!\, c"] \arrow[r] & \mathcal{M}_N \end{tikzcd}$$

Given
$$\begin{tikzcd} S \arrow[r, "c"] \arrow[dr] & \mathcal{M}_{N,p} \arrow[d, "pr"] \\ & \mathcal{M}_N \end{tikzcd}$$
$u := pr \circ c$

Then $c \longmapsto C \subseteq u^* \mathcal{E}[p]$

$u \longmapsto u^*(\mathcal{E}, \alpha^{univ}) / S$

constructing $(u^*\mathcal{E}, u^* \alpha^{univ}, C \subseteq u^* \mathcal{E})$

<u>Examples</u> 0) $G = \underline{\Gamma}_S$ constant

$\mathrm{Sub}_d(G) = \underline{\mathrm{Sub}_d(\Gamma)}_S$

1) $k = \bar{k}$ char $k = p$ $\quad \alpha_p := \ker\left(\begin{matrix} \mathbb{A}^1_k \to \mathbb{A}^1_k \\ z \mapsto z^p \end{matrix} \right)$

$$\mathrm{Sub}_p(\alpha_p^{\oplus r})(k) = \mathbb{P}^{r-1}(k)$$

Reason Classification of order p group schemes:

Any $H \subset \alpha_p^r$ of order p is $\cong \alpha_p$.

Combine with $\mathrm{Hom}(\alpha_p, \alpha_p^r) = k^r$.

2) E/k EC $k = \bar{k}$ char $k = p$

$$\mathrm{Sub}_p(E[p])(k) = \begin{cases} \{\cong \mu_p, \cong \mathbb{Z}/p\} & E \text{ ordinary} \\ \{\mathrm{Spec}\, \mathcal{O}_E / m_E^p\} & E \text{ supersing.} \end{cases}$$

$\Rightarrow E[p] \cong \mu_p \times \mathbb{Z}/p$

Cor $\mathcal{M}_{N,p} \to \mathcal{M}_N$

is surjective + quasi-finite,

hence finite (since also projective).

Contrast $\mathbb{F}_p \otimes \mathcal{M}_{pN} = \emptyset$

3) $k = \bar{k}$ char $k = p$ S/k

$H \in \mathrm{Sub}_p(\mathbb{Z}/p \times \mu_p)(S)$

H defines a decomposition.

$S = S_0 \amalg S_1$ where

$s \in S_0 \iff H(s) = \{0\} \times \mu_p$.

Above S_1, $H \cap (\{1\} \times \mu_p) \subseteq \mu_p$

defines section $h : S_1 \to \mu_p$

Determines H fully : $H \cap (\{i\} \times \mu_p)$

$= i \cdot h(S_1)$

Conversely, any $h : S \to \mu_p$ defines

$$H = \bigcup_{i \in \mathbb{Z}/p} \{i\} \times i \cdot h(S)$$

$\Longrightarrow$ $\mathrm{Sub}_p(\mathbb{Z}/p \times \mu_p) \cong \mathrm{Spec}\, k \amalg \mu_p$

In ptic, of degree $p+1$ / $\mathrm{Spec}\, k$.

Rmk This computes

$$\begin{tikzcd} \operatorname{Spec} k \times_{\mathcal{M}_N} \mathcal{M}_{N,p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_{N,p} \arrow[d] \\ \operatorname{Spec} k \arrow[r, "{(E,\alpha)}"] & \mathcal{M}_N \end{tikzcd}$$

$k = \bar{k}$, $\operatorname{char} k = p$

E ordinary